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INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES
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MANAGEMENT

PROVED FIXED POINT THEOREMS FOR WEAK CONTRACTIONS IN A NON-NORMAL CONE METRIC SPACE

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ABSTRACT

Various fixed point theorems for compatible mappings satisfying contractive type conditions and assuming continuity of at least one of the mappings, have been obtained by many authors. It has been known that there exist maps that have a discontinuity in the domain but which have fixed points, moreover, the maps involved in every case were continuous at the fixed point. In this paper we prove fixed point theorems for weak contractions in a non-normal cone metric space.

KEYWORDS: Fixed point, weak contraction, cone metric space

INTRODUCTION

Let (E, τ) be a topological vector space and P a subset of E, P is called a cone if
1. P is closed, non-empty and P □ $\{0\}$,
2. $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b,
3. $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, a partial ordering $\leq$ with respect to P is defined by $x \leq y$ if and only if $y - x \in P$, $x < y$ if $x \leq y$ and $x \leq y$, while $x << y$ will
stand for $y - x \in int\, P$, $int\, P$ denotes the the interior of P. If E is a normed space, then the
cone P is called normal (with respect to this norm) if there is a number$M > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\| x \| \leq M \| y \|$.

The least positive integer satisfying this norm inequality is called the normal constant of P [2]. Of course, there are non-normal cones [1-5].

Definition 1.1

Let (X, d) be a cone metric space, $x \in X$ and $\{xn\}^{\infty}_{n=1}$ a sequence in X.
Then
1. $\{xn\}^{\infty}_{n=1}$ converges to x whenever for every $c \in E$ with $0 << c$ there is a natural number N such that $d(xn, x) << c$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} xn = x$ or $xn \to x$.
2. $\{xn\}^{\infty}_{n=1}$ is a Cauchy sequence whenever for every $c \in E$ with $0 << c$ there is a natural number N such that $d(xn, xm) << c$ for all $n,m \geq N$.
3. (X, d) is a complete cone metric space if every Cauchy sequence is convergent. Recently, the following results were obtained.

Theorem 1.1 Let (X, d) be a complete cone metric space and P be a normal cone with normal constant K. Suppose the mapping $T : X \to X$ satisfies the contractive condition
$d(Tx,Ty) \leq kd(x, y)$, for all $x, y \in X$,
where $k \in [0, 1)$ is a constant. Then T has a unique fixed point in X. For any $x \in X$, iterative sequence $\{Tnx\}\infty$ $n=1$ converges to the fixed point.

Theorem 1.2. Let (X, d) be a complete cone metric space and the mapping $T : X \to X$ satisfy the contractive condition $d(Tx,Ty) \leq kd(x, y)$, for all $x, y \in X$, where $k \in [0, 1)$ is a constant. Then T has a unique fixed point in X. For each $x \in X$, the iterative sequence $\{Tnx\}\infty$ $n=1$ converges to the fixed point.

Theorem 1.3. Let (X, d) be a complete cone metric space and the mapping $T : X \to X$ satisfy the contractive condition
$d(Tx,Ty) \leq k(d(Tx,x) + d(Ty, y))$, (1) ; for all $x, y \in X$,where $k \in [0, 1/2)$ is a constant. Then T has a unique fixed point in X. For each $x \in X$, the iterative sequence $\{T^n x\}^{\infty}_{n=1}$ converges to the fixed point.

Theorem 1.4 Let (X, d) be a complete cone metric space and the mapping $T : X \to X$ satisfy the contractive condition

$d(Tx,Ty) \leq k(d(Tx, y) + d(x, Ty))$, (2) for all $x, y \in X$, where $k \in [0, 1/2)$ is a constant. Then T has a unique fixed point in X. For each $x \in X$, the iterative sequence $\{Tnx\}\infty$ $n=1$ converges to the fixed point.
In this paper we use the definition of the weak contraction mappings due to Berinde on cone metric spaces and prove some fixed point theorems of weak contractions. It is worth mentioning that the class of weak contractions includes the classes [6].

MAIN RESULTS

Definition 2.1 Let (X, d) be a complete cone metric space. A map $T : X \to X$ is called a weak contraction if there exists a constant $a \in (0, 1)$ and some $b \geq 0$ such that
$d(Tx,Ty) \leq ad(x, y) + bd(y, Tx)$ for all $x, y \in X$. (3)
Theorem 2.1 Let (X, d) be a complete cone metric space and the mapping $T : X \to X$ a weak contraction. Then T has a fixed point in X.

Proof:
For each $x0 \in X$ and $n \geq 1$, let $x1 = Tx0$, and $xn+1 = Txn = Tn+1x0$. Then
$d(xn, xn+1) = d(Txn-1, Txn) \leq ad(xn-1, xn) + bd(xn, Txn-1) = ad(xn-1, xn) \leq a2d(xn-2, xn-1) \leq \dots \leq and(x0, x1)$. So for $n > m$, $d(xm, xn) \leq (a^m + a^{m+1} + \dots + a^{n-1})d(x0, x1) \leq a^m/ 1 - a(d(x0, x1)$.
Let $0 << c$ be given. Choose a natural number N such that $a^m/ 1 - a(d(x0, x1) << c$ for every $m \geq N$. Thus $d(xm, xn) \leq a^m/ 1 - a(d(x0, x1) <<$ for every $n > m \geq N$.
Therefore the sequence $\{xn\}\infty$ $n=1$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exists $z \in X$ such that $xn \to z$. Choose a natural number $N1$ such that $d(xn, z) \leq c / 2(1+b)$ for every $n \geq N1$
Hence for $n \geq N1$ we have $d(z, Tz) \leq d(z, xn+1) + d(xn+1, Tz) = d(z, xn+1) + d(Txn, Tz)$
$\leq d(z, xn+1) + ad(xn, z) + bd(z, Txn)$
$= (1+b)d(z, xn+1) + ad(xn, z)$
$\leq (1 + b)d(z, xn+1) + d(xn, z)$
$\leq (1 + b)[d(z, xn+1) + d(xn, z)]$
$<< (1+b) [\{c/ 2(1+b)\} + \{c/ 2(1+b)\}] = c$ for every $n \geq N1$.
Thus
$d(z, Tz) << c/m$ for all $m \geq 1$.
So
$c/m - d(z, Tz) \in P$ for all $m \geq 1$. Since
$c/m \to 0$(as $m \to \infty$), and $P$is closed, $-d(z, Tz) \in P$. But $d(z, Tz) \in P$. Therefore $d(z, Tz) = 0$ and so
$Tz = z$.
We now show that Theorem 1.3 and Theorem 1.4 are corollaries of our results.
Corollary 2.1. Let (X, d) be a complete cone metric space. Any mapping $T : X \to X$ satisfying the contractive condition (1) is a weak contraction and so has a fixed point.
Proof. We follow By (1), we have,
$d(Tx,Ty) \leq k(d(x, Tx) + d(y, Ty))$
$\leq k\{[d(x, y) + d(y, Tx)] + [d(y, Tx) + d(Tx,Ty)]\}$,
which implies,
$(1 - k)d(Tx,Ty) \leq kd(x, y) + 2kd(y, Tx)$,
and which yields,
$d(Tx,Ty) \leq k/(1 - k)d(x, y) +\{2k/(1 - k)\}d(y, Tx)$ for all $x, y \in X$. Since $0 < k < 1/2$, (3) holds with $a = k/1 - k$, and $b = 2k/1 - k$

.**Corollary 2.2.** Let (X, d) be a complete cone metric space. Any mapping $T : X \to X$ satisfying the contractive condition (2) is a weak contraction and so has a fixed point.

Proof. We follow [1]. Using $d(x, Ty) \leq d(x, y) + d(y, Tx) + d(Tx,Ty)$,
by (2) we get
$d(Tx,Ty) \leq k[d(x, y) + d(y, Tx) + d(Tx,Ty) + d(y, Tx)]$
$(1 - k)d(Tx,Ty) \leq kd(x, y) + 2kd(y, Tx)$
$d(Tx,Ty) \leq k/(1 - k)d(x, y) +2k/(1 - k)d(y, Tx)$
which is (3) with $a = k/1 - k$, and $b = 2k/1 - k \geq 0$

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